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Hybrid Laplace transform technique for Stefan problems with radiation–convection boundary condition

HAN-TAW CHEN and JAE-YUH LIN
Department of Mechanical Engineering, National Cheng Kung University, Tainan, Taiwan 701, Republic of China

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Abstract—The hybrid application of the Laplace transform technique and the finite difference method (FDM) to one-dimensional Stefan problems involving the radiative and convective boundary condition is studied. The radiative term is linearized by Taylor's series approximation, and then the above hybrid method is used. This scheme is obtained by the use of the Laplace transform technique for the time-dependent terms and the fixed-grid FDM for space domain. It can be found from various illustrated examples that excellent agreement is obtained between the present results and those of early works. For the phase-change problem subjected to the nonlinear boundary condition, three or four iterations are required to obtain a convergent result at a specific time. The present analysis also demonstrates that the application of the Laplace transform technique is no longer limited to phase-change problems with the linear boundary condition.

INTRODUCTION

THE TRANSIENT heat conduction problem involving a change of phase due to melting or solidification is of particular interest in many industrial applications, such as in the casting of metals, the freezing of food and the making of ice, etc. The analytical solution of the phase-change problem is inherently difficult to determine because the location of the moving interface is not known *a priori*. However, the study of the phase-change problem has become a highly popular subject in recent years due to its vast applications in the areas of thermal energy storage or metallurgical solidification. An extensive review of the methods for solving the phase-change problems is given in ref. [1]. A few exact solutions of phase-change problems are currently available [2, 3]. These exact solutions are restricted to the problems of heat transfer in a semi-infinite region and are subjected to simple boundary or initial conditions. For most phase-change problems, approximate and numerical methods are commonly employed. Approximate solutions are practical when great accuracy is not required. Various approximate techniques, such as the heat balance integral method, the variational method, the perturbation method and the series expansion method, have been proposed. A brief discussion of these approximate methods is given in ref. [1].

Numerical methods are more practical in solving the phase-change problem. Based on the choice of the dependent variables used in the energy conservation equation, the numerical formulations can be classified into two main categories. The first formulation is regarded as the temperature-based method. In this general classical method, the temperature is the dependent variable and energy conservation equations are written respectively for the solid and liquid regions. The major difficulty of this formulation is the representation of the discontinuity of the temperature gradient at the liquid–solid interface. To overcome this difficulty, the second formulation, which uses the enthalpy or the apparent heat capacity as a dependent variable, is employed. However, the application of the enthalpy formulation to the phase-change problem often causes a serious numerical instability in the temperature distribution near the phase-change boundary as well as at the position of the boundary itself, because an enthalpy discontinuity exists at the location of the liquid–solid interface $s(t)$ [4]. The present study applies the temperature-based formulation to analyze the phase-change problem. As was mentioned above, the major difficulty of this formulation is that the temperature gradient at $x = s(t)$ is discontinuous and $s(t)$ is unknown *a priori*.

The present study uses the FDM in space domain and the Laplace transform technique for the time-dependent terms to analyze the phase-change problem. The present solution marches in time. This solution at a specific time interval is obtained by employing the inputs from the previous time step. In the present method, $s(t)$ at a given specific time t_i is regarded as constant, and then the length $s(t)$ is taken at the increased value for the next time interval. In other words, $s(t)$ at the previous time step is used to determine the new location of the moving interface. Successive iteration at a specific time interval is also used to correct the inputs from the previous time step. This procedure is followed until the boundary

NOMENCLATURE

Bi	Biot number, hb/k
b	reference length
c	specific heat of the material
Er	radiation variable, $=h\sigma\varepsilon\theta_m^3/k$
$\{f\}$	global force vector
h	heat transfer coefficient
$[K]$	global thermal conductance matrix in the transformed domain
k	thermal conductivity
L	latent heat of the material
l	distance between two nodes
n	total number of nodes
p	Laplace transform parameter
Q	surface heat flux
q	dimensionless surface heat flux, $Qb/k\theta_m$
S	location of the solid–liquid interface
St	Stefan number, $c\theta_m/L$
s	dimensionless location of the solid–liquid interface, S/b
T	dimensionless temperature
T_a	dimensionless ambient temperature, θ/θ_a
$\tilde{T}$	previously iterated temperature
t	dimensionless time, $k\tau/b^2\rho c$
u	suggested heat flux at the interface
X	space coordinate
x	dimensionless space coordinate, X/b.

Greek symbols

ε	emissivity of the material
θ	temperature
θ_a	ambient temperature
θ_e	environment temperature
θ_m	melting temperature
λ	parameter to determine the value of s
ρ	density of the material
σ	Stefan–Boltzmann constant
τ	time.

conditions at the interface are satisfied. In this computational procedure, the value of $(\partial T_s/\partial x)|_{x=s(t_i)}$ is first assumed to be constant at a given specific time interval $[t_i-\Delta t_i, t_i]$ while solving the phase-change problem. Then, $s(t_i)$ can be predicted. The above computations must be performed from $t=0$ initially. To the best of our knowledge, only Tamma and Railkar [5] and Ku and Chan [6] have applied the Laplace transform technique to analyze the phase-change problem. Tamma and Railkar [5] introduced the transfinite element methodology in conjunction with the enthalpy formulation for a phase-change problem. It can be seen from Fig. 13 in ref. [5] that the prediction of $s(t)$ did not agree well with the exact solution. Moreover, Tamma and Railkar [5] also did not investigate the nonlinear phase-change problem. Ku and Chan [6] proposed a generalized Laplace transform technique to obtain a closed-form solution for linear phase-change problems. Similarly, as stated in their work [6], their method also had a severe limitation to the nonlinear problems caused by temperature-dependent thermal properties or by nonlinear boundary conditions.

The one-dimensional phase-change problem in a finite region subjected to radiative and convective boundary conditions has been solved by Chung and Yeh [7] and Yan and Huang [8]. They respectively applied Biot's variational method and Goodman's integral technique [7] and the regular perturbation method [8] to analyze the above problem. In the present investigation, the present hybrid method is extended to such a problem arising from aerodynamic and radiative cooling or heating. Both the temperature distribution and the location of the moving interface are to be determined. As will be seen later, the location of the moving interface obtained by the present method agrees with that of Chung and Yeh [7].

MATHEMATICAL FORMULATION

A semi-infinite solid initially at its solidification (or melting) temperature T_m is confined to a half-space ($X>0$). The governing equation for the temperature distribution in the liquid region can be written as

$$\rho c\frac{\partial\theta}{\partial\tau}=k\frac{\partial^2\theta}{\partial X^2},\quad \text{in}\quad 0<X<S(\tau),\quad \tau>0 \tag{1}$$

where $S(\tau)$ is the location of the solid–liquid interface.

At time $\tau=0$, the boundary surface at $X=0$ is subjected to the following boundary condition and is maintained at that condition for $\tau>0$:

$$-k\frac{\partial\theta}{\partial X}=Q(\tau)+h(\theta_a-\theta)+\sigma\varepsilon(\theta_e^4-\theta^4)\quad \text{at}\quad X=0 \tag{2}$$

where h is the heat transfer coefficient, $Q(\tau)$ denotes the imposed surface heat flux, σ is the Stefan–Boltzmann constant and ε is the emissivity of the material. For simplicity, both the ambient temperature θ_a and the environmental temperature θ_e are assumed to be equal in the present analysis.

The coupling conditions at the interface $X=S(\tau)$ are

$$\theta(X,\tau)=\theta_m\quad \text{at}\quad X=S(\tau),\quad \tau>0 \tag{3a}$$

$$-k\frac{\partial\theta}{\partial X}=\rho L\frac{\mathrm{d}S}{\mathrm{d}\tau} \quad \text{at} \quad X=S(\tau), \quad \tau>0 \tag{3b}$$

where L is the latent heat of the fusion.

The initial conditions are

$$\theta(X,\tau)=\theta_{\mathrm{m}} \quad \text{for} \quad \tau=0, \quad \text{in} \quad X>0 \tag{4a}$$

$$S(\tau)=0 \quad \text{for} \quad \tau=0. \tag{4b}$$

We introduce the dimensionless variables

$$T=\frac{\theta}{\theta_{\mathrm{m}}}, \quad T_{\mathrm{a}}=\frac{\theta_{\mathrm{a}}}{\theta_{\mathrm{m}}}, \quad x=\frac{X}{b}$$

$$t=\frac{k\tau}{b^2\rho c}, \quad s=\frac{S}{b}, \quad Bi=\frac{hb}{k}, \quad q=\frac{Qb}{k\theta_{\mathrm{m}}}$$

$$Er=\frac{b\sigma\varepsilon\theta_{\mathrm{m}}^3}{k} \quad \text{and} \quad St=\frac{c\theta_{\mathrm{m}}}{L}$$

where b denotes an arbitrary reference length. Inserting these dimensionless variables into equations (1)–(4) leads to the following dimensionless differential equations:

$$\frac{\partial T}{\partial t}=\frac{\partial^2 T}{\partial x^2}, \quad \text{in} \quad 0<x<s(t), \quad t>0 \tag{5a}$$

$$-\frac{\partial T}{\partial x}=q(t)+Bi\,(T_{\mathrm{a}}-T)+Er\,(T_{\mathrm{a}}^4-T^4) \quad \text{at} \quad x=0 \tag{5b}$$

$$T(x,t)=1 \quad \text{at} \quad x=s(t), \quad t>0 \tag{5c}$$

$$-\frac{\partial T}{\partial x}=\frac{1}{St}\frac{\mathrm{d}s}{\mathrm{d}t} \quad \text{at} \quad x=s(t), \quad t>0 \tag{5d}$$

$$T(x,t)=1 \quad \text{and} \quad s(t)=0 \quad \text{for} \quad t=0 \tag{5e}$$

where St is the Stefan number, which signifies the importance of sensible heat relative to the latent heat.

The linearized form of equation (5b) using Taylor's series approximation [9] is

$$-\frac{\partial T}{\partial x}=q(t)+Bi\,(T_{\mathrm{a}}-T)+Er\,(T_{\mathrm{a}}^4-4\bar{T}^3T+3\bar{T}^4) \quad \text{at} \quad x=0 \tag{5f}$$

where $\bar{T}$ denotes the previously iterated temperature.

A brief illustration of the present hybrid method has been described in our previous works [9, 10]. To avoid duplication, its computational procedures are not presented in the present study.

In the present study the value of $(\partial T/\partial x)|_{x=s(t)}$ is assumed to be constant at a given time interval, i.e.

$$-\left.\frac{\partial T}{\partial x}\right|_{x=s(t)}=u \quad \text{in} \quad t_i\leqslant t\leqslant t_i+\Delta t_i, \quad i=1,2,\ldots \tag{6}$$

where $t_1=0$, $\Delta t_i=t_{i+1}-t_i$ and u is constant at a specific time interval and is determined by iteration. Thus the moving boundary velocity, $\mathrm{d}s/\mathrm{d}t$, at a specific time interval can be written as:

$$\frac{\mathrm{d}s}{\mathrm{d}t}=St\cdot u. \tag{7}$$

To estimate the location of the moving interface, a simple finite difference approximation is used, i.e.

$$\frac{\mathrm{d}s}{\mathrm{d}t}(t)\simeq\frac{s(t_{i+1})-s(t_i)}{\Delta t}, \quad t_i<t\leqslant t_i+\Delta t_i, \quad i=1,2,\ldots. \tag{8}$$

Substituting equation (8) into equation (7) yields the value of $s(t_{i+1})$ as:

$$s(t_{i+1})=s(t_i)+u\cdot St\cdot\Delta t \tag{9}$$

where $s(t_i)$ is the location of the moving interface at $t=t_i$.

The value of u is unknown for any time interval and is evaluated by iteration. However, once the value of u is determined, the location of the solid–liquid interface will be obtained from equation (9). In the present study, two different values of u at a specific time interval are guessed arbitrarily, and then the Secant method is applied to determine a new guessed value of u. This computational procedure is repeated until the boundary conditions at the location of the solid–liquid interface are satisfied.

SOLUTION METHOD

The Laplace transforms of equations (5) and (6) are respectively

$$\frac{\partial^2\tilde{T}}{\partial x^2}=p\tilde{T}-1 \quad 0<x<s(t_s) \tag{10}$$

and

$$-\frac{\partial\tilde{T}}{\partial x}=\tilde{q}+Bi\left(\frac{T_{\mathrm{a}}}{p}-\tilde{T}\right)+Er\left[\frac{1}{p}(T_{\mathrm{a}}^4+3\bar{T}^4)-4\bar{T}^3\tilde{T}\right] \quad \text{at} \quad x=0 \tag{11a}$$

$$-\frac{\partial\tilde{T}}{\partial x}=\frac{u}{p} \quad \text{at} \quad x=s(t_s) \tag{11b}$$

$$\tilde{T}(x,p)=\frac{1}{p} \quad \text{for} \quad x=s(t_s) \tag{11c}$$

where t_s denotes a specific time.

The Laplace transform of a function $\phi(x,t)$ is defined as

$$\tilde{\phi}(x,p)=\int_0^\infty \mathrm{e}^{-pt}\,\phi(x,t)\,\mathrm{d}t \tag{12}$$

where p is the Laplace transform parameter.

The discretized forms of equations (10) and (11), using the central difference approximation, are given as

$$\frac{1}{l^2}(\tilde{T}_{i-1}-2\tilde{T}_i+\tilde{T}_{i+1})=p\tilde{T}_i-1 \quad i=1,\ldots,n \tag{13}$$

$$-\frac{1}{2l}(\tilde{T}_2-\tilde{T}_0)=\tilde{q}+Bi\left(\frac{T_a}{p}-\tilde{T}_1\right)+Er\left[\frac{1}{p}(T_a^4+3\tilde{T}_1^4)-4\tilde{T}^3\tilde{T}_1\right] \quad (14a)$$

$$-\frac{1}{2l}(\tilde{T}_{n+1}-\tilde{T}_{n-1})=\frac{u}{p} \quad \text{at} \quad x=s(t_s) \quad (14b)$$

$$\tilde{T}_n=\frac{1}{p} \quad (14c)$$

where l denotes the distance between two nodes and is uniform in the present study.

The arrangement of equations (13) and (14) gives the following vector–matrix form as

$$[K]\{\tilde{T}\}=\{f\} \quad (15)$$

where

$$[K]=\begin{bmatrix} B_1C_1 & & & \\ A_2B_2C_2 & & & \\ & & & \\ & & A_{n+1}B_{n+1}C_{n+1} & \\ & & & A_nB_n \end{bmatrix} \quad (16a)$$

$$\{\tilde{T}\}^{\mathrm{T}}=\{\tilde{T}_1\tilde{T}_2\ldots\tilde{T}_{n-1}\tilde{T}_n\} \quad (16b)$$

$$\{f\}^{\mathrm{T}}=\{f_1f_2\ldots f_{n-1}f_n\} \quad (16c)$$

$$B_1=-2-pl^2-2l\,Bi-8Er\,l\tilde{T}_1^3 \quad (16d)$$

$$C_1=2 \quad (16e)$$

$$f_1=-2l\tilde{q}-\frac{2l}{p}T_a\,Bi-\frac{2l}{p}Er\,(T_a^4+3\tilde{T}_1^4) \quad (16f)$$

$$A_i=C_i=1, \quad i=2,3,\ldots,n-1 \quad (16g)$$

$$B_i=-2-pl^2, \quad i=2,3,\ldots,n-1 \quad (16h)$$

$$f_i=-l^2, \quad i=2,3,\ldots,n-1 \quad (16i)$$

$$A_n=2 \quad (16j)$$

$$B_n=-2-pl^2 \quad (16k)$$

$$f_n=\frac{2l}{p}u-l^2. \quad (16l)$$

In equation (15), the thermal conductance matrix $[K]$ is an $(n\times n)$ band matrix with complex numbers, $\{\tilde{T}\}$ is an $(n\times 1)$ vector representing the unknown transformed temperatures and the thermal load $\{f\}$ is an $(n\times 1)$ vector representing the forcing terms.

In the work of Rubinsky and Cravahlo [11] for the phase-change problem, the temperature distribution must be found at time $(t_i+\Delta t_i)$ in order to continue the iteration procedure. Since their scheme [11] is not self-starting, the location of the solid–liquid interface at the first time step must be found by using Neumann's exact solution for a semi-infinite medium. In fact, the exact solutions for the phase-change problems are limited to only a few simple cases. As will be seen later, the present study does not need to perform this procedure. In the present method, a specific time t_s and the value of $(\partial T/\partial x)|_{x=s(t_s)}$ are initially assumed, where $t_s=t_i+\Delta t_i$, $i=1,2,\ldots$. Then, an explicit prediction of $s(t_s)$ can be obtained from equation (9). Furthermore, solving equation (15) determines the temperature at the interface $T_c(s(t_s),t_s)$. A newly guessed value of u is updated by using the Secant method. This single procedure is iterated until the equality of the boundary condition at the moving interface is satisfied within 10^{-4}. These results will be applied to produce the predictions of u and $s(t)$ at the next time step.

ILLUSTRATED EXAMPLES

In all of the computations, the step sizes $\Delta x=s(t_s)/10$ and $\Delta t=t_s/100$ are used.

Example 1

A liquid initially at its melting temperature θ_m ($\theta_m>0$) is confined to a half-space ($x>0$). For time $t>0$, the temperature of the boundary surface at $x=0$ is kept at a constant temperature $\theta=0$. The exact solutions for the temperature distribution in the solid region and the location of the moving interface are [2]

$$T=\frac{\theta}{\theta_m}=\frac{\mathrm{erf}\,(x/2\sqrt{t})}{\mathrm{erf}\,(\lambda)} \quad (17)$$

$$s(t)=2\lambda\sqrt{t} \quad (18)$$

where the exact value of λ is determined from the following transcendental equation:

$$\sqrt{\pi}\lambda\,\mathrm{e}^{\lambda^2}\,\mathrm{erf}\,(\lambda)=St. \quad (19)$$

Table 1 and Fig. 1 show a comparison of λ for various St values for the exact solution, the present solution, Goodman's integral solution [12] and the coupled integral solution [13]. For this problem two or three iterations are required to obtain the present

Table 1. Comparison of λ for various St values

	λ			
St	Exact	Present	Integral [12]	Coupled integral [13]
0.020	0.1	0.09990	0.09999	0.09975
0.082	0.2	0.20225	0.20224	0.20044
0.191	0.3	0.30855	0.30730	0.30191
0.356	0.4	0.41855	0.41528	0.40429
0.592	0.5	0.52841	0.52583	0.50778
0.920	0.6	0.63550	0.63746	0.61154
1.373	0.7	0.73970	0.74902	0.71489
1.996	0.8	0.84130	0.85846	0.81595
2.858	0.9	0.94000	0.96426	0.91295
4.060	1.0	1.03640	1.06482	1.00372
5.755	1.1	1.13115	1.15880	1.08624
8.172	1.2	1.22485	1.24499	1.15877
11.663	1.3	1.31800	1.32265	1.22035
16.776	1.4	1.41140	1.39144	1.27080
24.370	1.5	1.50520	1.45139	1.31073
35.817	1.6	1.60055	1.50286	1.34130

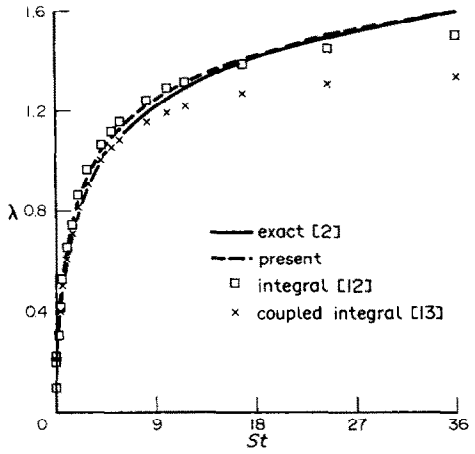

FIG. 1. Comparison of λ for various St values.

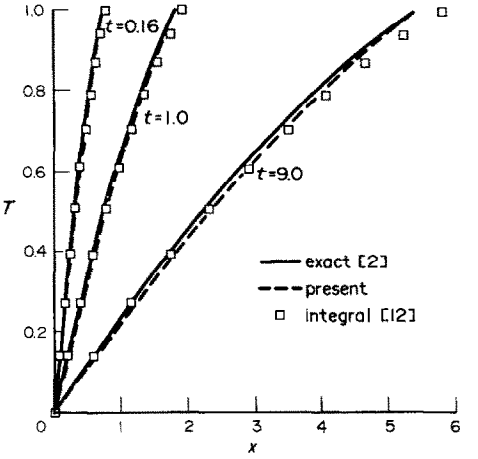

FIG. 3. The temperature distribution in the solid region at various selected times when $St = 0.858$.

solution at each time step. It is seen that the present solution agrees with the exact solution. The maximum difference of λ is found to be within 6% between the analytical solutions and the present solutions. However, this difference for λ will be reduced if a suitable time step Δt can be chosen. Further comparison is also made with the numerical results by using the method of Murray and Landis [14] for the location of the moving interface against time, as shown in Fig. 2. Excellent agreement is found between the present results and those of Murray and Landis. It can also be found that both Goodman's integral solution [12] and the coupled integral solution [13] depart from the exact solution for larger St values. Figures 3 and 4 show a comparison of the temperature distribution in the solid region for various St values at various selected times for the exact solution, the present solution and Goodman's integral solution. It is seen that the present solution is in good agreement with the exact solution over all ranges of the Stefan number. However, the difference of the temperature distribution between the exact solution and Goodman's integral solution increases with St and t, especially near the location of the solid–liquid interface.

Example 2

This example considers the melting of a semi-infinite bar initially in the solid phase and at its solidification temperature. The melting starts at $t = 0$ due to a dissipated heat flux across the boundary $x = 0$, i.e.

$$\frac{\partial T}{\partial x} = -\mathrm{e}^{t} \quad \text{at} \quad x = 0, \quad t > 0. \tag{20}$$

Furzeland [15] has determined the analytical solution of this problem for the temperature distribution in the liquid region and the location of the moving

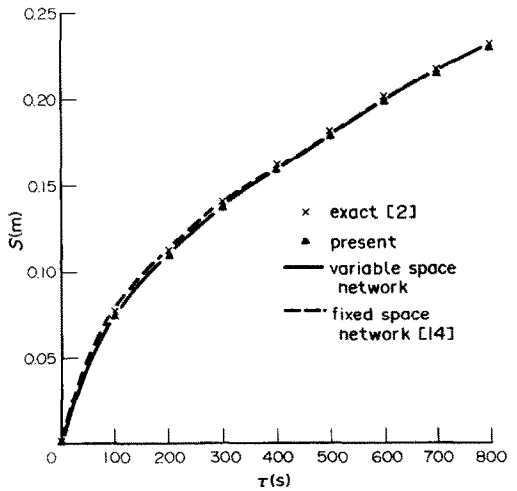

FIG. 2. Comparison between the present results and those of ref. [14] for the location of the moving interface with time.

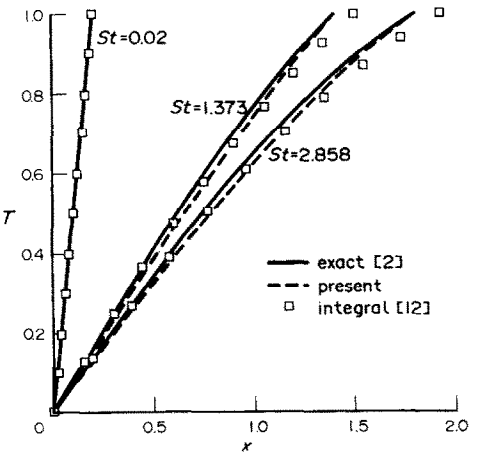

FIG. 4. The temperature distribution in the solid region for various St values at $t = 1$.

Table 2. Comparison of $s(t)$ at various selected times when $St = 1$

t	$s(t)$ Exact [15]	Present	Relative error (%)
0.1	0.1	0.1000	
0.2	0.2	0.2000	
0.5	0.5	0.4997	0.06
1.0	1.0	0.9966	0.34
1.5	1.5	1.4868	0.88
2.0	2.0	1.9690	1.55

Table 3. Comparison of $s(t)$ at various selected times when $St = 0.2$

t	$s(t)$ Exact [16]	Present	Relative error (%)
0.1	0.01996	0.01998	0.100
0.5	0.09903	0.09951	0.485
1.0	0.19624	0.19802	0.907
1.5	0.29178	0.29567	1.333
2.0	0.38578	0.39236	1.706

interface when $St = 1.0$. These results are respectively given as follows:

$$T(x, t) = e^{t - x} \tag{21}$$

and

$$s(t) = t. \tag{22}$$

Table 2 shows a comparison of $s(t)$ between the exact solutions and the present solutions. The temperature distribution in the liquid region at various selected times when $St = 1$ is shown in Fig. 5. It can be found from Table 2 and Fig. 5 that the present hybrid method has good accuracy for linear phase-change problems.

Example 3

Example 3 considers the same conditions as Example 2 above, except that the bar is subjected to a constant heat flux condition at $x = 0$ as

$$\frac{\partial T}{\partial x} = -1 \quad \text{at} \quad x = 0, \quad t > 0. \tag{23}$$

Lozano and Reemtsen [16] have presented a closed-form solution of this problem. Their results are applied to test the accuracy of the present method. The comparison between their results [16] and the present solutions is shown in Table 3 when $St = 0.2$. It can be seen that the present method can still yield considerable accuracy for Example 3.

Example 4

The problem of predicting solidification rates has been widely used in many fields ranging from freeze-drying to metal casting. When the solidification takes place at high temperature, such as in the case of metal castings, radiative and convective heat transfer are both significant mechanisms of cooling at the fixed surroundings. To further show the accuracy of the present hybrid method for such problems, a problem subjected to the nonlinear boundary condition at $x = 0$ is studied. The boundary condition at $x = 0$ dissipates heat by convection and radiation into a medium at θ_a. This example is the same as Example 1, except that the liquid is subjected to the nonlinear boundary condition at $x = 0$. Thus the boundary condition at $x = 0$ can be written as

$$\frac{\partial T}{\partial x} = Bi \cdot (T - T_a) + Er \cdot (T^4 - T_a^4). \tag{24}$$

Three or four iterations are required to obtain a convergent result at any selected time when the present method is applied to solve Example 4. The obtained results are shown in Figs. 6 and 7. The temperature distribution in the solid region at any selected time for various Er and T_a values is shown in Fig. 6 when $St = Bi = 1$. Figure 6 shows that the surface temperature at $x = 0$ gradually decreases with increasing time and Er. In the absence of an exact solution in the existing literature, approximate solutions obtained by

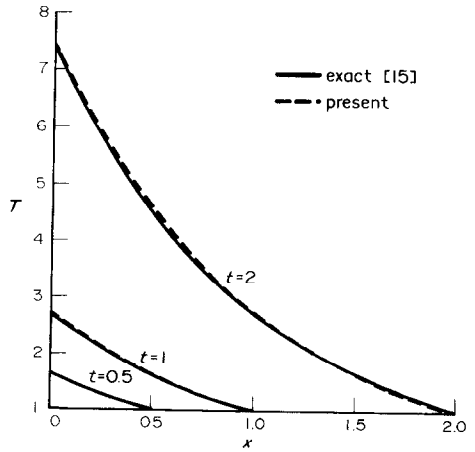

FIG. 5. The temperature distribution in the liquid region at various selected times when $St = 1$.

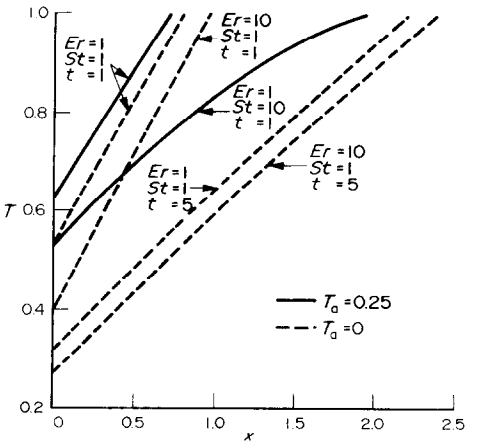

FIG. 6. The temperature distribution in the solid region at various selected times when $Bi = 1$ and $St = 1, 10$.

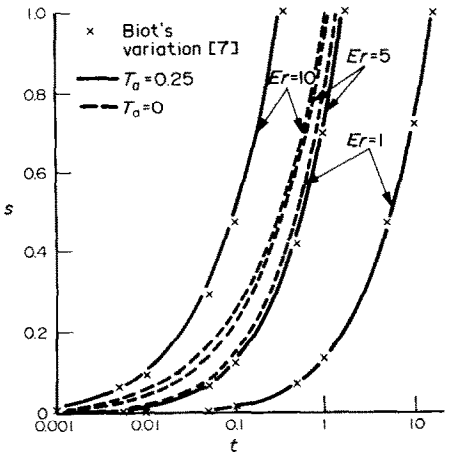

FIG. 7. Variation of $S(t)$ with t for various Er values when $Bi = St = 1$.

Biot's variational method and the heat balance integral technique were found and compared with the present solutions. Results shown in Fig. 2 of ref. [7] are not plotted in Fig. 6. However, a comparison of Fig. 2 in ref. [7] and Fig. 6 shows that the temperature history at the radiation and convection surface obtained by the present method agrees with that predicted by Biot's variational method when $Er = St = t = Bi = 1$. Further comparison is made for $Er = Bi = t = 1$ and $St = 10$. It can be seen that the difference between them will become great for the case of $St = 10$. However, excellent agreement can be obtained between the present result and that predicted by the heat balance integral method. This discrepancy results from the temperature profile chosen in the solid region. It is clear that the temperature profile in the solid region for $Er = Bi = t = 1$ and $St = 10$ is not a linear approximation. Figure 7 shows the variation of $s(t)$ with the dimensionless time for various Er values when $St = Bi = 1$. The solidification rate can easily be evaluated from the slope of the curves shown in Fig. 7. In addition, Fig. 7 also shows that the present results are in good agreement with those obtained by Biot's variational method. This comparison also implies that the present results for $s(t)$ agree with those of Goodling and Khader [17] obtained by an FDM. The above comparisons show that the application of the Laplace transform technique to such problems is no longer limited to phase-change problems subjected to linear boundary conditions.

CONCLUSIONS

The present study introduces a new numerical treatment for one-dimensional phase-change problems. Both the temperature distribution in the solid (or liquid) region and the location of the moving interface are obtained. It is seen from some illustrated examples that the present hybrid method has good accuracy even for the phase-change problem with the nonlinear boundary condition. For this problem with the nonlinear boundary condition, Taylor's series approximation must first be used to linearize the radiative term. In the present analysis, the hybrid method is applied only to analyze the problem of melting (or solidification) in a semi-infinite slab. However, it is sufficiently general for extension to more involved phase-change problems. Analysis involving temperature-dependent properties is underway. A similar technique can be used for one-dimensional phase-change problems in spheres and cylinders.

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